

# On the convergence to the multiple Wiener-Itô integral

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## Abstract

We study the convergence to the multiple Wiener-Itô integral from processes with absolutely continuous paths. More precisely, consider a family of processes, with paths in the Cameron-Martin space, that converges weakly to a standard Brownian motion in  $\mathcal{C}_0([0, T])$ . Using these processes, we construct a family that converges weakly, in the sense of the finite dimensional distributions, to the multiple Wiener-Itô integral process of a function  $f \in L^2([0, T]^n)$ . We prove also the weak convergence in the space  $\mathcal{C}_0([0, T])$  to the second order integral for two important families of processes that converge to a standard Brownian motion.

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## 1 Introduction and preliminaries

Let  $Y$  be a semimartingale with trajectories belonging to the space  $\mathcal{D}([0, T])$  of functions right continuous with left limits in all point in  $[0, T]$ , and define the following iterated Itô integrals

$$J_k(Y)_t = \begin{cases} Y_t & \text{if } k = 1 \\ \int_0^t J_{k-1}(Y)_{s-} dY_s & \text{for } k \geq 2. \end{cases}$$

Suppose that  $\{X^\varepsilon\}_{\varepsilon>0}$  is a family of semimartingales with paths in  $\mathcal{D}([0, T])$  that converges weakly in this space to another semimartingale  $X$ , as  $\varepsilon$  tends to zero. Avram (1988) proved that the following statements are equivalent

$$\begin{aligned} \mathcal{L}(X^\varepsilon, [X^\varepsilon, X^\varepsilon]) &\xrightarrow{w} \mathcal{L}(X, [X, X]) \quad \text{when } \varepsilon \downarrow 0, \quad \text{and} \\ \mathcal{L}(J_1(X^\varepsilon), \dots, J_m(X^\varepsilon)) &\xrightarrow{w} \mathcal{L}(J_1(X), \dots, J_m(X)) \quad \text{when } \varepsilon \downarrow 0, \end{aligned}$$

(here  $\xrightarrow{w}$  denotes the weak convergence in  $\mathcal{D}([0, T])^2$  and  $\mathcal{D}([0, T])^m$  respectively) where, if we denote by  $Y^c$  the continuous part of a semimartingale  $Y$ , the process  $[Y, Y]$  is defined by

$$[Y, Y]_t = \langle Y^c, Y^c \rangle_t + \sum_{s \leq t} (\Delta Y_s)^2.$$

This result shows that in order to obtain (joint) weak convergence of Itô multiple integrals we need the convergence of  $X^\varepsilon$  to  $X$  but also also the convergence of the second order variations. When our semimartingale is the Wiener process, there is a lot of important examples of families of processes with absolutely continuous paths converging in law in  $\mathcal{C}([0, T])$  to it. In this case, clearly, we do not have convergence of the quadratic variations to that of the Brownian motion.

Consider the Cameron-Martin space:

$$\mathcal{H} := \{\eta \in \mathcal{C}([0, T]) : \eta_t = \int_0^t \eta'_s ds, \eta' \in L^2([0, T])\},$$

and a family of processes  $(\eta_\varepsilon)_{\varepsilon > 0}$  with paths belonging to the Cameron Martin space given by

$$\eta_\varepsilon(t) = \int_0^t \theta_\varepsilon(s) ds, \quad (1)$$

such that  $(\eta_\varepsilon)_{\varepsilon > 0}$  converges weakly to a standard Brownian motion in  $\mathcal{C}_0([0, T])$ , the space of continuous functions defined in  $[0, T]$  which are null at zero.

Consider now, for a function  $f \in L^2([0, T]^n)$ , the multiple integrals

$$I_{\eta_\varepsilon}(f)_t = \int_0^t \cdots \int_0^t f(t_1, \dots, t_n) d\eta_\varepsilon(t_1) \cdots d\eta_\varepsilon(t_n).$$

In Bardina and Jolis (2000) the convergence in law of  $(I_{\eta_\varepsilon}(f))_\varepsilon$  was studied. The authors proved that in order to obtain convergence for all families  $(\eta_\varepsilon)$  with values in  $\mathcal{H}$  and converging in law to the Wiener process, the function  $f$  needs to be given by a multimeasure. For other classes of functions, only partial results were obtained with some particular families  $(\eta_\varepsilon)$ . In all the cases the limit was the Stratonovich integral of  $f$  with respect to the Wiener process, as defined by Solé and Utz (1990). This fact is not surprising, taking into account that the multiple Stratonovich integral must satisfy the rules of the ordinary calculus. On the other hand, this integral is a complicated object, it is defined by a limiting procedure and only some classes of functions (as tensor products or continuous functions) are recognized as Stratonovich integrable.

A natural question is that of the possibility of obtaining, for a function  $f$ , its multiple Wiener-Itô-type integral as a limit in law of some multiple integrals with respect to the absolutely continuous processes  $\eta_\varepsilon$ . Since in the definition of the multiple Wiener-Itô integral with respect to the Wiener process, the approximating procedure implies the suppression of the values on the diagonals, one can expect that a similar idea will allow to obtain this integral as a limit law.

We denote by  $Y_{\eta_\varepsilon}^f$  the stochastic processes defined by

$$\begin{aligned} Y_{\eta_\varepsilon}^f(t) &:= \int_{[0, t]^n} f(x_1, x_2, \dots, x_n) \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} d\eta_\varepsilon(x_1) \cdots d\eta_\varepsilon(x_n) \\ &= \int_{[0, t]^n} f(x_1, x_2, \dots, x_n) \prod_{i=1}^n \theta_\varepsilon(x_i) \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} dx_1 \cdots dx_n, \end{aligned} \quad (2)$$

for all  $t \in [0, T]$ . We have studied the weak convergence of the finite dimensional distributions of  $(Y_{\eta_\varepsilon}^f)_{\varepsilon > 0}$  to that of the corresponding multiple Wiener-Itô-integral of  $f$  with respect to the Wiener

process, and also the convergence in  $\mathcal{C}_0([0, T])$  of second order integrals for two important families of process  $(\eta_\varepsilon)$ . With regard to the convergence of the finite dimensional distributions, we have proved that there is convergence under rather general conditions on  $(\eta_\varepsilon)$ , see Theorem 2.3. For the convergence in  $\mathcal{C}_0([0, T])$  of the second order integral, we have proved it for the so-called Donsker and Kac-Stroock families of processes. It is worth to note that in all the results obtained here, the function  $f$  is an arbitrary function in  $L^2([0, T]^n)$ , that is, all the domain of the Wiener-Itô integral. This is a very different situation from that of Bardina and Jolis (2000).

Section 2 deals with the problem of the convergence of finite dimensional distributions and Section 3 is devoted to prove convergence in the space of continuous functions for the second order integral with respect to the Donsker and Kac-Stroock processes. In all the paper we denote the positive multiplicative constants that do not depend neither on  $\varepsilon$  nor on the function  $f$  by  $C$ , although their values can change from an expression to another one.

## 2 Convergence of the finite dimensional distributions

### 2.1 Some general results

We first state a general lemma that will be the main tool in order to prove the convergence of the finite dimensional distributions. We state it in our particular setting.

**Lemma 2.1** *Let  $(S, \|\cdot\|)$  be a normed space and consider  $\{J^\varepsilon\}_{\varepsilon \geq 0}$  a family of linear applications defined on  $S$  with values in the space of  $m$ -dimensional finite a.s. random variables,  $(L^0(\Omega))^m$ . Denote by  $|\cdot|$  the Euclidian norm in  $\mathbb{R}^m$ .*

*Assume that there exists a positive constant  $C$  such that for all  $f \in S$*

$$\sup_{\varepsilon \geq 0} E |J^\varepsilon(f)| \leq C \|f\|. \quad (3)$$

*Assume also that there exists a dense subset  $D \subset S$  such that for all  $f \in D$ ,  $J^\varepsilon(f)$  converges in law to  $J^0(f)$ , when  $\varepsilon$  tends to 0.*

*Then,  $J^\varepsilon(f)$  converges in law to  $J^0(f)$ , for all  $f \in S$ , when  $\varepsilon$  tends to 0.*

We will denote by  $\mathcal{E}'^n$  the space of simple functions on  $[0, T]^n$  that can be written as

$$f(x_1, \dots, x_n) = \sum_{k=1}^m \alpha_k I_{\Delta_k}(x_1, \dots, x_n), \quad (4)$$

where,  $m \in \mathbb{N}$ ,  $\alpha_k \in \mathbb{R}$ , for all  $k \in \{1, \dots, m\}$ , and  $\Delta_k = (a_k^1, b_k^1] \times (a_k^2, b_k^2] \times \dots \times (a_k^n, b_k^n]$  with  $[a_k^h, b_k^h] \cap [a_k^l, b_k^l] = \emptyset$  for all  $h \neq l$ .

**Lemma 2.2** *Let  $(\eta_\varepsilon)_{\varepsilon > 0}$  be a family of processes with trajectories in the Cameron-Martin space  $\mathcal{H}$  of the form (1). Assume that the finite dimensional distributions of the family  $(\eta_\varepsilon)_{\varepsilon > 0}$  converge in law to the finite dimensional distributions of a standard Brownian motion  $W$  when  $\varepsilon$  tends to 0.*

*Consider  $f \in \mathcal{E}'^n$ . Then, the finite dimensional distributions of the processes  $Y_{\eta_\varepsilon}^f$  defined in (2) converge in law to the finite dimensional distributions of the multiple Wiener-Itô integral  $I_n(f \cdot I_{[0, t]^n})$  when  $\varepsilon$  tends to 0.*

*Proof:* Since  $f \in \mathcal{E}'^n$  it follows that, for  $\varepsilon$  small enough,

$$f(x_1, x_2, \dots, x_n) \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} = f(x_1, x_2, \dots, x_n),$$

for all  $(x_1, x_2, \dots, x_n) \in [0, T]^n$ . And then, if  $f$  is given by (4), for those  $\varepsilon$ ,

$$Y_{\eta_\varepsilon}^f(t) = \sum_{k=1}^m \alpha_k \prod_{i=1}^n [\eta_\varepsilon(b_k^i \wedge t) - \eta_\varepsilon(a_k^i \wedge t)].$$

We conclude from the convergence of the finite dimensional distributions of  $\eta_\varepsilon$  to that of the Brownian motion that for all  $t_1, \dots, t_r \in [0, T]$  the vector  $(Y_{\eta_\varepsilon}^f(t_1), \dots, Y_{\eta_\varepsilon}^f(t_r))$  converges in law to

$$\left( \sum_{k=1}^m \alpha_k \prod_{i=1}^n [W(b_k^i \wedge t_1) - W(a_k^i \wedge t_1)], \dots, \sum_{k=1}^m \alpha_k \prod_{i=1}^n [W(b_k^i \wedge t_r) - W(a_k^i \wedge t_r)] \right),$$

when  $\varepsilon$  tends to 0.

But, since  $f \in \mathcal{E}'^n$ , by the definition of the multiple Wiener-Itô integral (see Itô, 1951) the last random vector equals to

$$(I_n(f \cdot I_{[0, t_1]^n}), \dots, I_n(f \cdot I_{[0, t_r]^n})).$$

□

The following theorem is the main result of this section and gives sufficient conditions for the family  $(\eta_\varepsilon)$  in order to have the convergence of the finite dimensional distributions of  $Y_{\eta_\varepsilon}^f$  to those of the multiple Wiener-Itô integral process, for any  $f \in L^2([0, T]^n)$ .

**Theorem 2.3** *Let  $(\eta_\varepsilon)_{\varepsilon > 0}$  be a family of processes with trajectories in the Cameron-Martin space  $\mathcal{H}$  of the form (1). Assume that the finite dimensional distributions of the family  $(\eta_\varepsilon)_{\varepsilon > 0}$  converge in law to the finite dimensional distributions of a standard Brownian motion when  $\varepsilon$  tends to 0. Assume also that there exists a positive constant  $C$  such that*

$$\sup_{\varepsilon > 0, t \in [0, T]} E |Y_{\eta_\varepsilon}^f(t)| \leq C \|f\|_{L^2([0, T]^n)}, \quad (5)$$

for all  $f \in L^2([0, T]^n)$ .

Then, the finite dimensional distributions of the family of processes  $\{Y_{\eta_\varepsilon}^f\}_{\varepsilon > 0}$  converge in law to those of the multiple Wiener-Itô integral  $I_n(f \cdot I_{[0, t]^n})$  for all  $f \in L^2([0, T]^n)$ , when  $\varepsilon$  tends to 0.

*Proof:* Take  $t_1, \dots, t_r \in [0, T]$ . In order to see that for all  $f \in L^2([0, T]^n)$ , the random vector  $(Y_{\eta_\varepsilon}^f(t_1), \dots, Y_{\eta_\varepsilon}^f(t_r))$  converges in law to

$$(I_n(f \cdot I_{[0, t_1]^n}), \dots, I_n(f \cdot I_{[0, t_r]^n})),$$

when  $\varepsilon$  tends to 0, we will apply Lemma 2.1. Take  $S = L^2([0, T]^n)$  and consider, for all  $\varepsilon > 0$ , the linear operators

$$\begin{aligned} J^\varepsilon : L^2([0, T]^n) &\longrightarrow (L^0(\Omega))^r \\ f &\longrightarrow (Y_{\eta_\varepsilon}^f(t_1), \dots, Y_{\eta_\varepsilon}^f(t_r)), \end{aligned}$$

and, for  $\varepsilon = 0$ , the linear operator,

$$\begin{aligned} J^0 : L^2([0, T]^n) &\longrightarrow (L^0(\Omega))^r \\ f &\longrightarrow (I_n(f \cdot I_{[0, t_1]^n}), \dots, I_n(f \cdot I_{[0, t_r]^n})). \end{aligned}$$

Condition (3) of Lemma 2.1 is satisfied because, by hypothesis,

$$\sup_{\varepsilon > 0} E |J^\varepsilon(f)| \leq C \|f\|_{L^2([0, T]^n)},$$

and, on the other hand, it is well known that

$$E |J^0(f)| \leq C \|f\|_{L^2([0, T]^n)}.$$

By Lemma 2.2 we have that, for all  $f \in \mathcal{E}'^n$ ,  $J^\varepsilon(f)$  converges in law to  $J^0(f)$ . This fact completes the proof because  $\mathcal{E}'^n$  is a dense subset of  $L^2([0, T]^n)$ .  $\square$

We can also consider the problem of the vectorial convergence to multiple Wiener-Itô integrals in the sense of the finite dimensional distributions. Fix a natural number  $d \geq 2$  and consider  $d$  integers  $n_1, n_2, \dots, n_d \geq 1$ . Let  $f_k \in L^2([0, T]^{n_k})$  for  $k = 1, \dots, d$  and consider the sequence of stochastic processes with values in  $\mathbb{R}^d$

$$Z^\varepsilon(t) = (Y_{\eta_\varepsilon}^{f_1}(t), \dots, Y_{\eta_\varepsilon}^{f_d}(t)), \quad t \in [0, T]. \quad (6)$$

with  $Y_{\eta_\varepsilon}^{f_k}$ ,  $k = 1, \dots, d$ , defined by (2). We can prove the next theorem that shows the convergence, as  $\varepsilon \rightarrow 0$ , of the finite dimensional distributions of  $Z^\varepsilon$  to those of the vector of multiple Wiener-Itô integrals

$$Z(t) = (I_{n_1}(f_1 \cdot I_{[0, t]^{n_1}}), \dots, I_{n_d}(f_d \cdot I_{[0, t]^{n_d}})), \quad t \in [0, T]. \quad (7)$$

**Theorem 2.4** *Let  $(\eta_\varepsilon)_{\varepsilon > 0}$  be a family of stochastic processes with paths in  $\mathcal{H}$  that converges in the sense of the finite dimensional distributions to a standard Brownian motion. Let, for  $k = 1, \dots, d$ ,  $f_k \in L^2([0, T]^{n_k})$  and assume that condition (5) is satisfied by every  $n_k$ ,  $k = 1, \dots, d$ . Then the finite dimensional distributions of the vector  $Z^\varepsilon$  given by (6) converges as  $\varepsilon \rightarrow 0$  to those of the vector  $Z$  given by (7).*

*Proof:* The proof follows similar arguments to that of Theorem 2.3 and then omitted.  $\square$

## 2.2 Examples

We will give now two examples of families  $\eta_\varepsilon$  for whose the above Theorems can be applied.

### 2.2.1 Convergence for the Donsker kernels

Consider now the particular case

$$\theta_\varepsilon(s) := \frac{1}{\varepsilon} \sum_{k=1}^{\infty} \xi_k I_{[k-1, k)} \left( \frac{s}{\varepsilon^2} \right),$$

where  $\{\xi_k\}$  is a sequence of independent, identically distributed random variables satisfying  $E(\xi_1) = 0$  and  $\text{Var}(\xi_1) = 1$ .

The processes  $\theta_\varepsilon$  will be called Donsker kernels, because the convergence in law of  $\eta_\varepsilon(t) = \int_0^t \theta_\varepsilon(s) ds$  to the Brownian motion in  $\mathcal{C}([0, T])$  is given by the well-known Donsker's Invariance Principle.

In view of Theorem 2.3, in order to prove the convergence of the finite dimensional distributions of  $\{Y_{\eta_\varepsilon}^f\}_\varepsilon$  to the finite dimensional distributions of  $I_n(f \cdot I_{[0,t]^n})$ , it is enough to prove that there exists some constant  $C > 0$  such that, for all  $f \in L^2([0, T]^n)$

$$\sup_{\varepsilon > 0, t \in [0, T]} E \left[ \int_{[0,t]^n} f(x_1, x_2, \dots, x_n) \prod_{i=1}^n \theta_\varepsilon(x_i) \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} dx_1 \cdots dx_n \right]^2 \leq C \|f\|_{L^2([0,T]^n)}^2.$$

We can assume, without loss of generality that  $f$  is symmetric. Notice that,

$$\begin{aligned} & E \left[ \int_{[0,t]^n} f(x_1, x_2, \dots, x_n) \prod_{i=1}^n \theta_\varepsilon(x_i) \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} dx_1 \cdots dx_n \right]^2 \\ &= \int_{[0,t]^{2n}} f(x_1, x_2, \dots, x_n) f(y_1, y_2, \dots, y_n) E \left[ \prod_{i=1}^n \theta_\varepsilon(x_i) \theta_\varepsilon(y_i) \right] \\ & \quad \times \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} I_{\{|y_i - y_j| > \varepsilon\}} dx_1 \cdots dx_n dy_1 \cdots dy_n. \end{aligned} \quad (8)$$

We can also suppose that  $\varepsilon < 1$ . In this case, the condition  $|x - y| > \varepsilon$  implies that  $|x - y| > \varepsilon^2$  and then,

$$\begin{aligned} & E \left[ \prod_{i=1}^n \theta_\varepsilon(x_i) \theta_\varepsilon(y_i) \right] \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} I_{\{|y_i - y_j| > \varepsilon\}} \\ &= E \left[ \frac{1}{\varepsilon^{2n}} \sum_{\substack{i_1, \dots, i_n, j_1, \dots, j_n \\ i_k \neq i_l, j_k \neq j_l \forall k \neq l}} \prod_{k=1}^n \xi_{i_k} \xi_{j_k} I_{[i_k - 1, i_k]} \left( \frac{x_k}{\varepsilon^2} \right) I_{[j_k - 1, j_k]} \left( \frac{y_k}{\varepsilon^2} \right) \right] \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} I_{\{|y_i - y_j| > \varepsilon\}}. \end{aligned}$$

Notice that the number of different indexes in each summand appearing in the above expression

is greater or equal than  $n$ . Therefore, using also the symmetry of  $f$ , we can write (8) as

$$\begin{aligned}
& \int_{[0,t]^{2n}} f(x_1, x_2, \dots, x_n) f(y_1, y_2, \dots, y_n) E \left[ \frac{1}{\varepsilon^{2n}} \sum' \prod_{k=1}^n \xi_{i_k} \xi_{j_k} I_{[i_k-1, i_k]} \left( \frac{x_k}{\varepsilon^2} \right) I_{[j_k-1, j_k]} \left( \frac{y_k}{\varepsilon^2} \right) \right] \\
& \times \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} I_{\{|y_i - y_j| > \varepsilon\}} dx_1 \cdots dx_n dy_1 \cdots dy_n \\
& + \int_{[0,t]^{2n}} f(x_1, x_2, \dots, x_n) f(y_1, y_2, \dots, y_n) E \left[ \frac{n!}{\varepsilon^{2n}} \sum_{\substack{i_1, \dots, i_n \\ i_k \neq i_l, \forall k \neq l}} \prod_{k=1}^n \xi_{i_k}^2 I_{[i_k-1, i_k]^2} \left( \frac{x_k}{\varepsilon^2}, \frac{y_k}{\varepsilon^2} \right) \right] \\
& \times \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} I_{\{|y_i - y_j| > \varepsilon\}} dx_1 \cdots dx_n dy_1 \cdots dy_n,
\end{aligned}$$

where  $\sum'$  denotes the sum over all the indexes satisfying that at least  $n+1$  among the  $i_1, \dots, i_n, j_1, \dots, j_n$  are different.

Using now that  $\{\xi_k\}$  is a sequence of independent, identically distributed random variables with  $E(\xi_1) = 0$  and  $\text{Var}(\xi_1) = 1$ , we can bound the last expression by

$$\begin{aligned}
& n! \int_{[0,t]^{2n}} |f(x_1, x_2, \dots, x_n) f(y_1, y_2, \dots, y_n)| \\
& \times \frac{1}{\varepsilon^{2n}} \left( \sum_{\substack{i_1, \dots, i_n \\ i_k \neq i_l, \forall k \neq l}} \prod_{k=1}^n I_{[i_k-1, i_k]^2} \left( \frac{x_k}{\varepsilon^2}, \frac{y_k}{\varepsilon^2} \right) \right) dx_1 \cdots dx_n dy_1 \cdots dy_n \\
& \leq n! \int_{[0,t]^n} f^2(x_1, x_2, \dots, x_n) \\
& \times \frac{1}{\varepsilon^{2n}} \left( \sum_{\substack{i_1, \dots, i_n \\ i_k \neq i_l, \forall k \neq l}} \prod_{k=1}^n I_{[i_k-1, i_k]} \left( \frac{x_k}{\varepsilon^2} \right) \int_{[0,t]^n} \prod_{k=1}^n I_{[i_k-1, i_k]} \left( \frac{y_k}{\varepsilon^2} \right) dy_1 \cdots dy_n \right) dx_1 \cdots dx_n \\
& \leq n! \int_{[0,t]^n} f^2(x_1, x_2, \dots, x_n) \left( \sum_{\substack{i_1, \dots, i_n \\ i_k \neq i_l, \forall k \neq l}} \prod_{k=1}^n I_{[i_k-1, i_k]} \left( \frac{x_k}{\varepsilon^2} \right) \right) dx_1 \cdots dx_n \\
& \leq n! \|f\|_{L^2([0,T]^n)}^2.
\end{aligned}$$

### 2.2.2 Convergence for the Kac-Stroock kernels

Consider now the following kernels introduced by Kac (1956)

$$\theta_\varepsilon(x) := \frac{1}{\varepsilon} (-1)^{N\left(\frac{x}{\varepsilon^2}\right)},$$

where  $N = \{N(s); s \geq 0\}$  is a standard Poisson process. Stroock (1982) proved that the family  $(\eta_\varepsilon)_{\varepsilon>0}$  with  $\eta_\varepsilon(t) = \int_0^t \theta_\varepsilon(s) ds$  converges in law in  $\mathcal{C}_0([0, T])$  to the Brownian motion.

As for the Donsker kernels, to prove the convergence of the finite dimensional distributions of  $\{Y_{\eta_\varepsilon}^f\}_\varepsilon$  to those of  $I_n(f \cdot I_{[0, t]^n})$ , it is enough to prove that there exists some constant  $C > 0$  such that, for all  $f \in L^2([0, T]^n)$

$$\sup_{\varepsilon>0, t \in [0, T]} E \left[ \int_{[0, t]^n} f(x_1, x_2, \dots, x_n) \prod_{i=1}^n \theta_\varepsilon(x_i) \prod_{\substack{i, j=1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} dx_1 \cdots dx_n \right]^2 \leq C \|f\|_{L^2([0, T]^n)}^2.$$

Observe that, denoting by  $\mathcal{P}_n$  the group of permutations of the set  $\{1, \dots, n\}$ , we have that

$$\begin{aligned} & E \left[ \prod_{i=1}^n \theta_\varepsilon(x_i) \theta_\varepsilon(y_i) \right] \prod_{\substack{i, j=1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} I_{\{|y_i - y_j| > \varepsilon\}} \\ &= E \left[ \prod_{i=1}^n \theta_\varepsilon(x_i) \theta_\varepsilon(y_i) \right] \prod_{\substack{i, j=1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} I_{\{|y_i - y_j| > \varepsilon\}} \sum_{\sigma, \psi \in \mathcal{P}_n} I_{\{x_{\sigma_1} \leq x_{\sigma_2} \leq \dots \leq x_{\sigma_n}\}} I_{\{y_{\psi_1} \leq y_{\psi_2} \leq \dots \leq y_{\psi_n}\}} \\ &= E \left[ \prod_{i=1}^n \theta_\varepsilon(x_i) \theta_\varepsilon(y_i) \right] \prod_{\substack{i, j=1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} I_{\{|y_i - y_j| > \varepsilon\}} \sum_{\sigma, \psi \in \mathcal{P}_n} I_{\{x_{\sigma_1}, y_{\psi_1} \leq \{x_{\sigma_2}, y_{\psi_2}\} \leq \dots \leq \{x_{\sigma_n}, y_{\psi_n}\}\}} \\ & \quad + E \left[ \prod_{i=1}^n \theta_\varepsilon(x_i) \theta_\varepsilon(y_i) \right] \prod_{\substack{i, j=1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} I_{\{|y_i - y_j| > \varepsilon\}} \sum_{\sigma, \psi \in \mathcal{P}_n} A(x_1, \dots, x_n, y_1, \dots, y_n; \sigma, \psi) \quad (9) \end{aligned}$$

where  $\{a, b\} \leq \{c, d\}$  means that  $a \vee b \leq c \wedge d$ , and where  $A(x_1, \dots, x_n, y_1, \dots, y_n; \sigma, \psi)$  is the sum of the indicator functions with all the other possible orders between the  $2n$  variables  $\{x_{\sigma_1} \leq x_{\sigma_2} \leq \dots \leq x_{\sigma_n}\}$  and  $\{y_{\psi_1} \leq y_{\psi_2} \leq \dots \leq y_{\psi_n}\}$ .

We will start with the first summand of the righthand side of (9). Notice that

$$\begin{aligned} & E \left[ \prod_{i=1}^n \theta_\varepsilon(x_i) \theta_\varepsilon(y_i) \right] I_{\{x_{\sigma_1}, y_{\psi_1} \leq \{x_{\sigma_2}, y_{\psi_2}\} \leq \dots \leq \{x_{\sigma_n}, y_{\psi_n}\}\}} \\ &= \frac{1}{\varepsilon^{2n}} E \left[ (-1)^{\sum_{i=1}^n \left( N\left(\frac{x_{\sigma_i}}{\varepsilon^2}\right) + N\left(\frac{y_{\psi_i}}{\varepsilon^2}\right) \right)} \right] I_{\{\{x_{\sigma_1}, y_{\psi_1}\} \leq \{x_{\sigma_2}, y_{\psi_2}\} \leq \dots \leq \{x_{\sigma_n}, y_{\psi_n}\}\}}. \end{aligned}$$

Using that for  $a, b \in \mathbb{N} \cup \{0\}$  we have  $(-1)^{a+b} = (-1)^{a-b}$ , the fact that the Poisson process has independent increments, and that if  $Z \sim \text{Poiss}(\lambda)$  then  $E[(-1)^Z] = \exp(-2\lambda)$ , we obtain that

the expectation appearing in the last expression is equal to

$$\exp \left( -2 \sum_{i=1}^n \left( \frac{|x_{\sigma_i} - y_{\psi_i}|}{\varepsilon^2} \right) \right).$$

Moreover,

$$\begin{aligned} & \int_{[0,t]^{2n}} |f(x_1, x_2, \dots, x_n) f(y_1, y_2, \dots, y_n)| \frac{1}{\varepsilon^{2n}} \exp \left( -2 \sum_{i=1}^n \left( \frac{|x_{\sigma_i} - y_{\psi_i}|}{\varepsilon^2} \right) \right) dx_1 \cdots dx_n dy_1 \cdots dy_n \\ & \leq \int_{[0,t]^n} f^2(x_1, x_2, \dots, x_n) \frac{1}{\varepsilon^{2n}} \left( \int_{[0,t]^n} \exp \left( -2 \sum_{i=1}^n \left( \frac{|x_{\sigma_i} - y_{\psi_i}|}{\varepsilon^2} \right) \right) dy_1 \cdots dy_n \right) dx_1 \cdots dx_n \\ & \leq \int_{[0,t]^n} f^2(x_1, x_2, \dots, x_n) dx_1 \cdots dx_n \\ & = \|f\|_{L^2([0,T]^n)}^2. \end{aligned}$$

We consider now the second summand of (9). We have showed that in the computation of the expectation is important the order of the  $2n$  variables  $\{x_{\sigma_1} \leq x_{\sigma_2} \leq \dots \leq x_{\sigma_n}\}$  and  $\{y_{\psi_1} \leq y_{\psi_2} \leq \dots \leq y_{\psi_n}\}$ . If we take the variables  $(x_1, \dots, x_n, y_1, \dots, y_n)$  in each summand of  $A(x_1, \dots, x_n, y_1, \dots, y_n; \sigma, \psi)$  in groups of two variables taking into account their order, necessarily one of the groups will be formed by two variables  $x_k, x_l$  (for some  $k \neq l \in \{1, \dots, n\}$ ). Then, when we compute the expectation the corresponding term will be

$$\exp \left( -2 \frac{|x_k - x_l|}{\varepsilon^2} \right)$$

and we have that

$$\frac{1}{\varepsilon^{2n}} \exp \left( -2 \frac{|x_k - x_l|}{\varepsilon^2} \right) I_{\{|x_k - x_l| > \varepsilon\}} \leq \frac{1}{\varepsilon^{2n}} e^{-\frac{2}{\varepsilon}} \leq C.$$

So, we have that

$$\begin{aligned} & \sum_{\sigma, \psi \in \mathcal{P}_n} \int_{[0,t]^{2n}} |f(x_1, x_2, \dots, x_n) f(y_1, y_2, \dots, y_n)| E \left[ \prod_{i=1}^n \theta_\varepsilon(x_i) \theta_\varepsilon(y_i) \right] \\ & \quad \times A(x_1, \dots, x_n, y_1, \dots, y_n; \sigma, \psi) \prod_{\substack{i, j = 1 \\ i \neq j}}^n I_{\{|x_i - x_j| > \varepsilon\}} I_{\{|y_i - y_j| > \varepsilon\}} dx_1 \cdots dx_n dy_1 \cdots dy_n \\ & \leq C \int_{[0,t]^{2n}} |f(x_1, x_2, \dots, x_n) f(y_1, y_2, \dots, y_n)| dx_1 \cdots dx_n dy_1 \cdots dy_n \\ & \leq C \|f\|_{L^2([0,T]^n)}^2. \end{aligned}$$

### 3 Convergence in law in $\mathcal{C}_0([0, T])$ of the second order integral

In this section we will see that in the case of the second order Wiener-Itô integral, for the examples introduced in Subsection 2.2, we can prove also the convergence in law in  $\mathcal{C}_0([0, T])$ .

Let us first mention that clearly for every  $\varepsilon > 0$  the paths of the process  $Y_{\eta_\varepsilon}^f$  are absolute continuous functions. On the other hand, since the multiple Wiener-Itô integrals can be expressed as iterate integrals

$$I_2(f \cdot I_{[0,t]^2}) = 2 \int_0^t \int_0^y f(x, y) dW(x) dW(y)$$

for any  $f \in L^2([0, T]^2)$ , the stochastic process  $(I_2(f \cdot I_{[0,t]^2}))_{t \geq 0}$  admits a version with continuous trajectories.

When  $n = 2$ , the processes  $Y_{\eta_\varepsilon}^f$  become

$$Y_{\eta_\varepsilon}^f(t) := \int_0^t \int_0^t f(x, y) \theta_\varepsilon(x) \theta_\varepsilon(y) I_{\{|x-y|>\varepsilon\}} dx dy, \quad (10)$$

where  $\theta_\varepsilon$  are the Kac-Stroock or the Donsker kernels. In this section we need more integrability for the variables  $\{\xi_k\}$  appearing in the Donsker kernels. Concretely we will assume that  $E(\xi_k)^4 < +\infty$ .

**Theorem 3.1** *Let  $f \in L^2([0, T]^2)$ . Then, the processes  $Y_{\eta_\varepsilon}^f$  given by (10) converge weakly to the multiple Wiener-Itô integral of order 2,  $I_2(f \cdot I_{[0,t]^2})$ , in the space  $\mathcal{C}_0([0, T])$  when  $\varepsilon$  tends to zero.*

*Proof:* We can assume without loss of generality that  $f$  is symmetric. We have proved, in the previous section, the convergence of the finite dimensional distributions. So, to prove the convergence in law, it is enough to prove that the family of laws of  $\{Y_{\eta_\varepsilon}^f\}_\varepsilon$  is tight in  $\mathcal{C}_0([0, T])$ .

It suffices to show that for  $s \leq t$

$$E(Y_{\eta_\varepsilon}^f(t) - Y_{\eta_\varepsilon}^f(s))^4 \leq C \left( \int_{[0,T]^2} \bar{f}^2(x, y) dx dy \right)^2, \quad (11)$$

where

$$\bar{f}(x, y) := f(x, y) I_{[0,t]^2}(x, y) - f(x, y) I_{[0,s]^2}(x, y).$$

Indeed, for  $s \leq t$

$$(I_{[0,t]^2} - I_{[0,s]^2})^2 = I_{[0,t]^2} - I_{[0,s]^2},$$

Therefore, if (11) is satisfied

$$\begin{aligned} E(Y_{\eta_\varepsilon}^f(t) - Y_{\eta_\varepsilon}^f(s))^4 &\leq C \left( \int_{[0,T]^2} \bar{f}^2(x, y) dx dy \right)^2 \\ &= C \left( \int_{[0,T]^2} f^2(x, y) (I_{[0,t]^2} - I_{[0,s]^2}) dx dy \right)^2 \\ &= C \left( \int_s^t \int_0^y f^2(x, y) dx dy + \int_s^t \int_0^x f^2(x, y) dy dx \right)^2 \\ &= C \left( \int_s^t \int_0^y f^2(x, y) dx dy \right)^2, \end{aligned}$$

using the symmetry of  $f$  in the last step. Then, by Billingsley criterium (see Theorem 12.3 of Billingsley (1968)), we will obtain tightness.

Notice that

$$\begin{aligned}
& E \left( Y_{\eta_\varepsilon}^f(t) - Y_{\eta_\varepsilon}^f(s) \right)^4 \\
&= \int_{[0,T]^8} \prod_{i=0}^3 \bar{f}(u_{2i+1}, u_{2i+2}) I_{\{|u_{2i+1} - u_{2i+2}| > \varepsilon\}} E \left( \prod_{i=1}^8 \theta_\varepsilon(u_i) \right) du_1 \cdots du_8 \\
&\leq \int_{[0,T]^8} \prod_{i=0}^3 |\bar{f}(u_{2i+1}, u_{2i+2})| I_{\{|u_{2i+1} - u_{2i+2}| > \varepsilon\}} \left| E \left( \prod_{i=1}^8 \theta_\varepsilon(u_i) \right) \right| du_1 \cdots du_8. \tag{12}
\end{aligned}$$

From now on we will study separately the Kac-Stroock case and the Donsker case.

### Proof of Theorem 3.1 for the Kac-Stroock kernels

In order to simplify notation denote by  $f^S$  the function defined as

$$f^S(u_1, \dots, u_8) = \sum_{\sigma \in \mathcal{P}_8} \prod_{i=0}^3 |\bar{f}(u_{\sigma_{2i+1}}, u_{\sigma_{2i+2}})| I_{\{|u_{\sigma_{2i+1}} - u_{\sigma_{2i+2}}| > \varepsilon\}}.$$

In the case of the Kac-Stroock kernels, using the same kind of computations that in Subsection 2.2.2, and using also the symmetry of  $f^S$ , we have that (12) can be bounded by

$$C \int_{[0,T]^8} \frac{1}{\varepsilon^8} f^S(u_1, \dots, u_8) I_{\{u_1 < u_2 < \dots < u_8\}} \prod_{i=0}^3 \exp \left( \frac{-2(u_{2i+2} - u_{2i+1})}{\varepsilon^2} \right).$$

Consider now the different summands appearing in the definition of  $f^S$ . Notice that if in a summand appears a factor of the type

$$\exp \left( \frac{-2(x - y)}{\varepsilon^2} \right) I_{\{x - y > \varepsilon\}}$$

we have that

$$\frac{1}{\varepsilon^8} \exp \left( \frac{-2(x - y)}{\varepsilon^2} \right) I_{\{x - y > \varepsilon\}} \leq \frac{1}{\varepsilon^8} e^{-\frac{2}{\varepsilon}} \leq C.$$

And so, all the terms with this type of factors can be bounded by

$$\begin{aligned}
C \int_{[0,T]^8} \prod_{i=0}^3 |\bar{f}(u_{2i+1}, u_{2i+2})| du_1 \cdots du_8 &= C \left( \int_{[0,T]^2} |\bar{f}(x, y)| dx dy \right)^4 \\
&\leq C \left( \int_{[0,T]^2} \bar{f}^2(x, y) dx dy \right)^2.
\end{aligned}$$

For the rest of summands appearing in  $f^S$ , we bound all the indicators by 1, and excepting symmetries, there are only two possible situations:

### Situation 1

We have terms of the type

$$\begin{aligned} & \int_{[0,T]^8} \frac{1}{\varepsilon^8} \prod_{i=1}^4 |\bar{f}(x_i, y_i)| \exp \left( \frac{-2|x_1 - x_2|}{\varepsilon^2} + \frac{-2|y_1 - y_2|}{\varepsilon^2} \right) \\ & \quad \times \exp \left( \frac{-2|x_3 - x_4|}{\varepsilon^2} + \frac{-2|y_3 - y_4|}{\varepsilon^2} \right) dx_1 \dots dx_4 dy_1 \dots dy_4. \end{aligned}$$

This kind of terms can be bounded by

$$\begin{aligned} & \int_{[0,T]^8} \frac{1}{\varepsilon^8} \bar{f}^2(x_1, y_1) \bar{f}^2(x_3, y_3) \exp \left( \frac{-2|x_1 - x_2|}{\varepsilon^2} + \frac{-2|y_1 - y_2|}{\varepsilon^2} \right) \\ & \quad \times \exp \left( \frac{-2|x_3 - x_4|}{\varepsilon^2} + \frac{-2|y_3 - y_4|}{\varepsilon^2} \right) dx_1 \dots dx_4 dy_1 \dots dy_4 \\ & + \int_{[0,T]^8} \frac{1}{\varepsilon^8} \bar{f}^2(x_2, y_2) \bar{f}^2(x_4, y_4) \exp \left( \frac{-2|x_1 - x_2|}{\varepsilon^2} + \frac{-2|y_1 - y_2|}{\varepsilon^2} \right) \\ & \quad \times \exp \left( \frac{-2|x_3 - x_4|}{\varepsilon^2} + \frac{-2|y_3 - y_4|}{\varepsilon^2} \right) dx_1 \dots dx_4 dy_1 \dots dy_4. \end{aligned}$$

Integrating, in the first summand of the last expression, with respect to  $x_2, y_2, x_4, y_4$  and in the second one with respect to  $x_1, y_1, x_3, y_3$  we have that the last expression is bounded by

$$\begin{aligned} & C \int_{[0,T]^4} \bar{f}^2(x_1, y_1) \bar{f}^2(x_3, y_3) dx_1 dy_1 dx_3 dy_3 + C \int_{[0,T]^4} \bar{f}^2(x_2, y_2) \bar{f}^2(x_4, y_4) dx_2 dy_2 dx_4 dy_4 \\ & = C \left( \int_{[0,T]^2} \bar{f}^2(x, y) dx dy \right)^2. \end{aligned}$$

### Situation 2

We have also terms of the type

$$\begin{aligned} & \int_{[0,T]^8} \frac{1}{\varepsilon^8} \prod_{i=1}^4 |\bar{f}(x_i, y_i)| \exp \left( \frac{-2|x_1 - x_2|}{\varepsilon^2} + \frac{-2|x_3 - x_4|}{\varepsilon^2} \right. \\ & \quad \left. + \frac{-2|y_1 - y_3|}{\varepsilon^2} + \frac{-2|y_2 - y_4|}{\varepsilon^2} \right) dx_1 \dots dx_4 dy_1 \dots dy_4. \end{aligned}$$

All these terms are bounded by

$$\begin{aligned}
& C \int_{[0,T]^8} \frac{1}{\varepsilon^8} \bar{f}^2(x_1, y_1) \bar{f}^2(x_4, y_4) \exp \left( \frac{-2|x_1 - x_2|}{\varepsilon^2} + \frac{-2|x_3 - x_4|}{\varepsilon^2} \right. \\
& \quad \left. + \frac{-2|y_1 - y_3|}{\varepsilon^2} + \frac{-2|y_2 - y_4|}{\varepsilon^2} \right) dx_1 \dots dx_4 dy_1 \dots dy_4 \\
& + C \int_{[0,T]^8} \frac{1}{\varepsilon^8} \bar{f}^2(x_2, y_2) \bar{f}^2(x_3, y_3) \exp \left( \frac{-2|x_1 - x_2|}{\varepsilon^2} + \frac{-2|x_3 - x_4|}{\varepsilon^2} \right. \\
& \quad \left. + \frac{-2|y_1 - y_3|}{\varepsilon^2} + \frac{-2|y_2 - y_4|}{\varepsilon^2} \right) dx_1 \dots dx_4 dy_1 \dots dy_4.
\end{aligned}$$

Integrating now, in the first summand of the last expression, with respect to  $x_2, y_2, x_3, y_3$  and in the second one with respect to  $x_1, y_1, x_4, y_4$  we have that the last expression is bounded by

$$\begin{aligned}
& C \int_{[0,T]^4} \bar{f}^2(x_1, y_1) \bar{f}^2(x_3, y_3) dx_1 dy_1 dx_3 dy_3 + C \int_{[0,T]^4} \bar{f}^2(x_2, y_2) \bar{f}^2(x_4, y_4) dx_2 dy_2 dx_4 dy_4 \\
& = C \left( \int_{[0,T]^2} \bar{f}^2(x, y) dx dy \right)^2.
\end{aligned}$$

### Proof of Theorem 3.1 for the Donsker kernels

Remember that in this case

$$\theta_\varepsilon(s) := \frac{1}{\varepsilon} \sum_{k=1}^{\infty} \xi_k I_{[k-1, k)} \left( \frac{s}{\varepsilon^2} \right),$$

where  $\{\xi_k\}$  is a sequence of independent, identically distributed random variables satisfying  $E(\xi_1) = 0$ ,  $\text{Var}(\xi_1) = 1$  and  $E(\xi_1)^4 < +\infty$ .

Remember also that we can assume that  $\varepsilon < 1$ , and then the condition  $|x - y| > \varepsilon$  implies that  $|x - y| > \varepsilon^2$ .

Expression (12) equals to

$$\begin{aligned}
& \int_{[0,T]^8} \prod_{i=1}^4 |\bar{f}(u_i, v_i)| I_{\{|u_i - v_i| > \varepsilon^2\}} |E(\prod_{i=1}^4 \theta_\varepsilon(u_i) \theta_\varepsilon(v_i))| du_1 \dots dv_4 \\
& = \int_{[0,T]^8} \prod_{i=1}^4 |\bar{f}(u_i, v_i)| I_{\{|u_i - v_i| > \varepsilon^2\}} \\
& \times \left| E \left( \sum_{\substack{i_1, \dots, i_4 \\ j_1, \dots, j_4}} \xi_{i_1} \dots \xi_{i_4} I_{[i_1-1, i_1]} \left( \frac{u_1}{\varepsilon^2} \right) \dots I_{[j_4-1, j_4]} \left( \frac{v_4}{\varepsilon^2} \right) \right) \right| du_1 \dots dv_4. \quad (13)
\end{aligned}$$

We have, on one hand, that the random variables  $\xi_i$  are independent with  $E(\xi_k) = 0$  and, on the other hand that

$$I_{\{|u_i - v_i| > \varepsilon^2\}} I_{[k-1, k)} \left( \frac{u_i}{\varepsilon^2} \right) I_{[k-1, k)} \left( \frac{v_i}{\varepsilon^2} \right) = 0.$$

Consequently, to compute the expectation in expression (13), we have to consider the different decompositions of 8 as sums of natural numbers between 2 and 4: (2+2+2+2), (2+2+4), (3+3+2)

and (4+4), that will be the exponents of the  $\xi_i$  in the products appearing in (13) with no null expectation. Taking into account that

$$\sum_k I_{[k-1,k]}(\frac{u}{\varepsilon^2}) I_{[k-1,k]}(\frac{v}{\varepsilon^2}) \leq I_{\{|u-v| < \varepsilon^2\}},$$

that  $E(\xi_i^4) < \infty$ , and doing similar computations to those of the last section, the expressions obtained with all these decompositions, except with the third one (3+3+2), can be bounded by

$$\frac{C}{\varepsilon^8} \int_{[0,T]^8} \prod_{i=0}^3 |\bar{f}(u_{2i+1}, u_{2i+2})| I_{\{|u_{2i+1} - u_{2i+2}| > \varepsilon^2\}} \sum_{\sigma \in \mathcal{P}_8} \prod_{i=0}^3 I_{\{|u_{\sigma_{2i+1}} - u_{\sigma_{2i+2}}| < \varepsilon^2\}} du_1 \dots du_8.$$

Observe that the products

$$\prod_{i=0}^3 |\bar{f}(u_{2i+1}, u_{2i+2})| I_{\{|u_{2i+1} - u_{2i+2}| > \varepsilon^2\}} \prod_{i=0}^3 I_{\{|u_{\sigma_{2i+1}} - u_{\sigma_{2i+2}}| < \varepsilon^2\}}$$

equal to zero for all permutation  $\sigma \in \mathcal{P}_8$  such that at least one of the sets of two variables  $\{u_1, u_2\}, \{u_3, u_4\}, \{u_5, u_6\}, \{u_7, u_8\}$  is transformed by  $\sigma$  in one of them. Then, one can only consider the permutations  $\sigma$  for which, given  $\{u_1, u_2\}, \{u_3, u_4\}, \{u_5, u_6\}, \{u_7, u_8\}$ , there exists always two couples among them such that their four variables are not paired in the product

$$\prod_{i=0}^3 I_{\{|u_{\sigma_{2i+1}} - u_{\sigma_{2i+2}}| < \varepsilon^2\}}.$$

Now, we can proceed as with the Kac-Stroock kernels. If, for instance, the two couples with the above property are  $\{u_1, u_2\}$  and  $\{u_3, u_4\}$  we majorize the product

$$\prod_{i=0}^3 |\bar{f}(u_{2i+1}, u_{2i+2})| I_{\{|u_{2i+1} - u_{2i+2}| > \varepsilon^2\}}$$

by

$$\frac{1}{2} \left( \bar{f}^2(u_1, u_2) \bar{f}^2(u_3, u_4) + \bar{f}^2(u_5, u_6) \bar{f}^2(u_7, u_8) \right)$$

and, for each summand, perform the integral first with respect to the remaining four variables. This allows to cancell the term,  $\frac{1}{\varepsilon^8}$  and we obtain the desired bound.

Finally, we must to study the term corresponding to the decomposition (3+3+2). Taking now into account that

$$\sum_k I_{[k-1,k]}(\frac{u}{\varepsilon^2}) I_{[k-1,k]}(\frac{v}{\varepsilon^2}) I_{[k-1,k]}(\frac{w}{\varepsilon^2}) \leq I_{\{\text{GD}\{u, v, w\} < \varepsilon^2\}},$$

where we denote by GD the greatest distance between a sequence of elements, the product of indicators that we will obtain in this case can be bounded by

$$\sum_{\sigma \in \mathcal{P}_8} I_{\{\text{GD}\{u_{\sigma_1}, u_{\sigma_2}, u_{\sigma_3}\} < \varepsilon^2\}} I_{\{\text{GD}\{u_{\sigma_4}, u_{\sigma_5}, u_{\sigma_6}\} < \varepsilon^2\}} I_{\{|u_{\sigma_7} - u_{\sigma_8}| < \varepsilon^2\}}.$$

Therefore, all the terms, excepting symmetries, will be of the form

$$f(x, y)f(s, t)f(u, v)f(z, w)I_{\{|x-y|>\varepsilon^2\}}I_{\{|s-t|>\varepsilon^2\}}I_{\{|u-v|>\varepsilon^2\}}I_{\{|z-w|>\varepsilon^2\}} \\ \times I_{\{\text{GD}\{x, s, u\}<\varepsilon^2\}}I_{\{\text{GD}\{y, t, z\}<\varepsilon^2\}}I_{\{|v-w|<\varepsilon^2\}}.$$

(Observe that we do not consider  $I_{\{\text{GD}\{x, s, u\}<\varepsilon^2\}}I_{\{\text{GD}\{y, t, v\}<\varepsilon^2\}}$  because in this case we obtain a factor  $I_{\{|z-w|<\varepsilon^2\}}I_{\{|z-w|>\varepsilon^2\}} = 0$ ).

This kind of term can be bounded by

$$f^2(x, y)f^2(z, w)I_A + f^2(s, t)f^2(u, v)I_B,$$

where

$$A := \{|v-w| < \varepsilon^2\} \cap \{|y-t| < \varepsilon^2\} \cap \{|x-s| < \varepsilon^2\} \cap \{|x-u| < \varepsilon^2\}$$

and

$$B := \{|x-s| < \varepsilon^2\} \cap \{|y-t| < \varepsilon^2\} \cap \{|z-t| < \varepsilon^2\} \cap \{|w-v| < \varepsilon^2\}.$$

Integrating with respect to  $u, s, t, v$  in the term corresponding to  $I_A$  and with respect to  $x, y, z, w$  in the term corresponding to  $I_B$ , we obtain the desired result.

The proof of Theorem 3.1 is now complete.  $\square$

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